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Minor-equivalence for infinite graphs

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Abstract

Two graphs are *minor-equivalent* if each is isomorphic to a minor of the other. In this paper, we give structural characterizations of the minor equivalence classes of the infinite full grid $G_{\mathbb{Z}\times\mathbb{Z}}$ and of the infinite half-grid $G_{\mathbb{Z}\times\mathbb{N}}$. A corollary of these results states that every minor of $G_{\mathbb{Z}\times\mathbb{Z}}$ that has a minor isomorphic to $G_{\mathbb{Z}\times\mathbb{N}}$ is minor-equivalent to one of $G_{\mathbb{Z}\times\mathbb{Z}}$ or $G_{\mathbb{Z}\times\mathbb{N}}$.

1. Introduction

Graphs in this paper may be finite or infinite. To simplify the notation, we shall consider only graphs with no loops and no multiple edges. However, all the results presented here can be easily extended to graphs in which loops and multiple edges are allowed.

Let G be a graph. We define a *minor* of G as follows. Suppose W is a set such that every element is a non-null connected (possibly infinite) subgraph of G, and no two such subgraphs meet. Moreover, suppose that F is a set every element of which is a connected (possibly infinite) subgraph of G meeting exactly two elements of W, and such that the intersection of every two distinct elements of F lies in some element of W. Then the graph with vertex set W, edge set F, and the obvious incidence relation is a *minor* of G. Conversely, if J is a minor of G, then the subgraph of G that is the union of all vertices and edges of J is the *expansion* of J in G, and is denoted by $\Sigma(J)$. We write $H \leqslant_{\mathrm{m}} G$ if H is isomorphic to a minor of G. If H is isomorphic to a minor of G, but G is not isomorphic to any minor of H, then we write $H <_{\mathrm{m}} G$. Two graphs G and H are *minor-equivalent*, written $G \cong_{\mathrm{m}} H$, if $G \leqslant_{\mathrm{m}} H$ and $H \leqslant_{\mathrm{m}} G$. We remark that it is easy to check that a graph H is isomorphic to a minor of a graph G if and only if H can be obtained from a subgraph of G by contracting some of

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its connected subgraphs where we note that these subgraph may be infinite. Thus our definition of a minor is equivalent to the one usually found in literature.

It is clear that minor-equivalence is indeed an equivalence relation. The equivalence class that contains a graph G will be denoted by $[G]_{\mathrm{m}}$. Obviously, if G is a finite graph, then $[G]_{\mathrm{m}}$ is equal to the isomorphism class containing G. In contrast, if G is infinite, then $[G]_{\mathrm{m}}$ may contain graphs from more than one isomorphism class. For example, if G is an infinite clique and H consists of G and a single isolated vertex, then $G \cong_{\mathrm{m}} H$, and thus $[G]_{\mathrm{m}} = [H]_{\mathrm{m}}$, even though G and H are not isomorphic. On the other hand, if G is a two-way-infinite path, then $[G]_{\mathrm{m}}$ is equal to the isomorphism class of G. Another interesting example of a graph whose minor-equivalence class coincides with its isomorphism class is a graph that is not isomorphic to any of its proper minors. The existence of such a graph has been shown in [2].

A graph G *covers* another graph H if $H <_{\mathrm{m}} G$ and, for every graph K such that $H \leqslant_{\mathrm{m}} K \leqslant_{\mathrm{m}} G$, either $K \cong_{\mathrm{m}} H$ or $K \cong_{\mathrm{m}} G$. The following is an immediate consequence of the definitions.

Proposition 1.1. *If G covers H, then G covers every graph that is minor-equivalent to H, and, similarly, H is covered by every graph that is minor-equivalent to G.*

If G is a finite non-null graph, then G covers some graphs, namely those that can be obtained from G by deleting a single edge, contracting a single edge, or deleting a single isolated vertex. Similarly, every finite graph is covered by another graph. However, if G is infinite, then G may or may not cover other graphs. It is easy to verify the following:

Proposition 1.2. *A graph that has a countably infinite vertex set and no edges covers no graphs.*

Proposition 1.3. *A two-way-infinite path covers the disjoint union of two one-way-infinite paths.*

A graph is *planar* if it has no minor isomorphic to K_5 or to $K_{3,3}$. We shall focus our attention on two infinite planar graphs: the full grid and the half-grid, defined as follows. The *full grid*, denoted by $G_{\mathbb{Z}\times\mathbb{Z}}$, has the set $\mathbb{Z}\times\mathbb{Z}$ as its vertex set, and two of its vertices $(i,j),(i',j')$ are joined by an edge if and only if $|i-i'|+|j-j'|=1$. The *half-grid*, denoted by $G_{\mathbb{Z}\times\mathbb{N}}$, is the graph obtained from $G_{\mathbb{Z}\times\mathbb{Z}}$ by deleting all the vertices whose second coordinate is negative. Two results of the paper, stated later as Theorems 4.1 and 5.1, give sufficient structural conditions for a graph to be minor-equivalent to, respectively, the infinite half-grid and the infinite full grid. As a corollary of these results, we prove the following:

Theorem 1.4. *The full grid covers the half-grid.*

Two of the many open problems that are related to Theorem 1.4 are as follows:

Question 1.5. Does the half-grid cover any graphs?

Question 1.6. Does the countably infinite clique cover any graphs?

Several other interesting questions arise by using the relation of topological embedding in place of the minor relation in such covering problems. Let G be a graph and let $G \times 2$ be the disjoint union of two copies of G. The graph G is *clonable* if $G \times 2 \leqslant_{\mathrm{m}} G$. It is easy to show the following:

Proposition 1.7. *If each of the graphs H and H' is covered by a clonable connected graph G, then $H \cong_{\mathrm{m}} H'$.*

Since both the half-grid and the countably infinite clique are clonable, it follows from Proposition 1.7 that, up to minor-equivalence, there is at most graph that is covered by the half-grid, and at most one graph that is covered by the countably infinite clique.

This section continues with some more terminology and notation, and then concludes with an outline of the remainder of the paper.

If E is a subset of the edge set of G, then $G \backslash E$ denotes the subgraph of G induced by the edges not in E. Similarly, if V is a subset of the vertex set of G, then $G - V$ denotes the subgraph of G induced by the vertices not in V. If H is a subgraph of G, then $G \backslash E(H)$ may be abbreviated as $G \backslash H$, and $G - V(H)$ may be abbreviated as $G - H$.

Suppose that H, K, and L are subgraphs of a graph G such that $H \cap K = \emptyset$. Assume also that L contains exactly one path that has one endvertex in each of H and K and is internally disjoint from $H \cup K$. Then this path will be denoted by $[H, K]_L$. If H has only one vertex h, then $[H, K]_L$ will be denoted by $[h, K]_L$. A similar convention applies when K has just one vertex.

A *tree* is a connected (possibly infinite) graph without cycles. A *rooted tree* is a pair (T, r) where T is a tree and r is one of its vertices. A *ray* is a tree that is a one-way-infinite path. A ray will always be considered to be rooted at its unique vertex of degree one. For two vertices u and v of a rooted tree (T, r), we shall write $u \leqslant_{(T,r)} v$, or simply $u \leqslant_T v$, if u is a vertex of $[r, v]_T$. If $u \leqslant_T v$ and, additionally, u and v are distinct, then we may write $u <_T v$. If ρ is a ray in G and K is a finite subgraph of G meeting ρ, then K_ρ denotes the maximal subpath of ρ that has both endvertices in K.

The tree that plays a special role in this paper is the *infinite binary tree*. Before formally introducing this tree, we need a few definitions on binary sequences.

A sequence is *binary* if all of its elements are in the set $\{0, 1\}$. For a non-negative integer n, the set of all binary sequences of length n will be denoted by $[2^n]$. The set of all finite binary sequences will be denoted by $[2^{<\omega}]$. If α and β are elements of $[2^{<\omega}]$, then $\alpha + \beta$ denotes the sequence that is the concatenation of α and β. In particular, $\alpha + 0$ is the sequence obtained by adjoining a single zero to the end of the sequence α. For a finite binary sequence α of positive length, the symbol α' will denote the sequence obtained from α by deleting its last element. For two binary sequences

α and β, we write $\alpha \prec \beta$ if the length of α is less than the length of β, or if their lengths are equal and α precedes β lexicographically. If $\alpha \prec \beta$ or $\alpha = \beta$, we shall write $\alpha \preccurlyeq \beta$. Clearly, the relation $\preccurlyeq$ is a linear order on the set of finite binary sequences. For a binary sequence α, let α^+ denote the successor of α in this relation.

The two binary sequences of length n that will be most frequently referred to in this paper are the one consisting of all zeros and the one consisting of all ones. These will be denoted by $\mathbf{0}_n$ and $\mathbf{1}_n$, respectively. If the value of n can be inferred from the context, then $\mathbf{0}_n$ and $\mathbf{1}_n$ will be abbreviated as $\mathbf{0}$ and $\mathbf{1}$. The binary sequence of length zero will be denoted by $\emptyset$.

The *infinite binary tree* T^{ω} is the graph with vertex set $[2^{<\omega}]$ in which all sets of the form $\{\alpha, \alpha+0\}$ or $\{\alpha, \alpha+1\}$ are edges. The infinite binary tree will be always considered to be rooted at the vertex $\emptyset$.

Two rays ρ and σ, which are subgraphs of the same graph G, are *equivalent* if, for every finite subgraph H of G, the infinite parts of $\rho - H$ and $\sigma - H$ lie in the same connected component of $G - H$. Halin [1] proved the following:

Theorem 1.8. *Two rays are equivalent if and only if there is another ray that meets both of them infinitely often.*

It is easy to verify that the above relation is an equivalence relation on rays which are subgraphs of a fixed infinite graph G. The equivalence classes of this relation are called the *ends* of G. An end is *thick* if it contains infinitely many pairwise-disjoint rays.

Suppose G is an infinite graph and $\mathscr{R}$ is a finite set of rays in G. A path P (or a cycle C) of G is *reduced* with respect to a ray ρ if P (or C) either intersects ρ along a path (perhaps consisting of one vertex only) or does not intersect it at all. A path P (or cycle C) *collates* a set $\mathscr{R}$ of rays of G if it meets all elements of $\mathscr{R}$ and is reduced with respect to every element of $\mathscr{R}$. A graph G is *round* if it satisfies conditions (C1)–(C3) below.

(C1) G is planar, connected, locally finite, and has exactly one end.

(C2) The end of G is thick.

(C3) For every finite subgraph H of G and every set $\mathscr{R}$ of pairwise disjoint rays of G such that $3 \leqslant |\mathscr{R}| < \infty$, there is a cycle in $G - H$ that collates $\mathscr{R}$.

The graph G is *flat* if it satisfies (C1) and (C2), but fails (C3).

It is straightforward to verify the following propositions.

Proposition 1.9. *The full grid is round.*

Proposition 1.10. *The half-grid is flat.*

Let J be a subgraph of G. A *vertex of attachment* of J in G is a vertex of J that is incident with an edge of G which is not an edge of J. A subgraph H is said to be *J-detached* if all vertices of attachment of H in G are in J. A *bridge* B of J in G is a subgraph of G satisfying the following three conditions.

(B1) B is not a subgraph of J.
(B2) B is J-detached in G.
(B3) No proper subgraph of B satisfies both (B1) and (B2).

The remainder of this paper is organized as follows. In Section 2, we investigate paths and cycles collating finite sets of pairwise-disjoint sets of rays. Section 3 contains two technical refinements of the well-known result of Robertson et al. stating that every finite planar graph is isomorphic to a minor of a sufficiently large grid. In Sections 4 and 5, we use the results of Sections 2 and 3 to prove theorems that may be viewed as inverses of Propositions 1.9 and 1.10. These results state that every flat graph is minor-equivalent to the half-grid, and every round graph is minor-equivalent to the full grid. As a consequence of these results, flat and round graphs satisfy Seymour's self-minor conjecture. More precisely, we have the following:

Corollary 1.11. *If G is flat or round, then G is isomorphic to a proper minor of itself.*

In Section 6, we use the results of the previous two sections to investigate subgraphs of $G_{\mathbb{Z}\times\mathbb{Z}}$. In Section 7, we employ the results of the previous section and the concept of planar duals to conclude the proof of Theorem 1.4.

2. Collating sets of rays

In this section, we shall study ways in which a path or a cycle can intersect a finite set of rays of a graph.

Lemma 2.1. *Let G be a locally finite planar graph with exactly one end, and let $\mathscr{R}$ be a finite set of pairwise-disjoint rays. Let H be a finite connected subgraph of G that meets all elements of $\mathscr{R}$. If P is a path meeting all elements of $\mathscr{R}$, then there is a path Q that is contained in $P \cup (\bigcup_{\rho\in\mathscr{R}} P_\rho)$ and collates $\mathscr{R}$.*

Proof. Without loss of generality, we may assume that no proper subpath of P meets all elements of $\mathscr{R}$. Suppose the vertices of P are $p_0, p_1, \ldots, p_m$ in the order listed. We define the path Q as follows. Let $s_0 = t_0 = p_0$. Assume that s_{i-1} and t_{i-1} have been defined as two vertices of P that lie on some ray ρ of $\mathscr{R}$ such that $t_{i-1} \neq p_m$, and $[t_{i-1}, p_m]_P - \{t_{i-1}\}$ avoids ρ. Let s_i be the vertex p_k of $[t_{i-1}, p_m]_P - \{t_{i-1}\}$ that meets some ray ρ' of $\mathscr{R}$ and whose index is as small as possible. Let t_i be the vertex p_k of $[s_i, p_m]_P$ that lies on ρ and whose index is as large as possible. Let h be the number such that $t_h = p_m$. For each $i \in \{0, 1, \ldots, h\}$, let ρ_i denote the element of $\mathscr{R}$ that contains both s_i and t_i. Finally, let

$$Q = \bigcup_{1\leqslant i\leqslant h} ([t_{i-1}, s_i]_P \cup [s_i, t_i]_{\rho_i}).$$

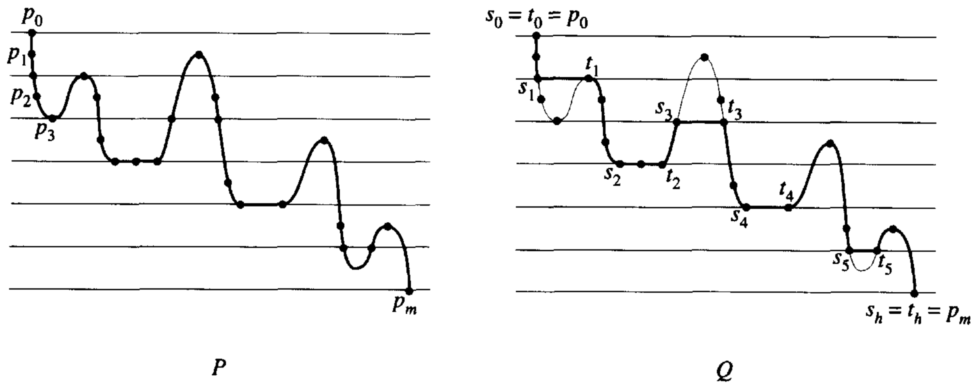

Fig. 1.

The process of constructing Q is illustrated in Fig. 1. It is clear from the construction that Q is reduced with respect to every element of $\mathscr{R}$ and that it is contained in $P \cup (\bigcup_{\rho \in \mathscr{R}} P_\rho)$. Thus, it remains to show that Q meets all elements of $\mathscr{R}$.

Suppose that Q avoids an element ρ of $\mathscr{R}$. Recall that P meets all elements of $\mathscr{R}$. Hence, there is a vertex p of $P \cap \rho$. It follows from the construction that there is a number $i \in \{1, 2, \ldots, h-1\}$ such that p lies on P between s_i and t_i. Let K be a finite subgraph of G that contains $H \cup P$ and all subpaths of elements of $\mathscr{R}$ with both end-vertices in $H \cup P$. Since the rays of $\mathscr{R}$ are in the same end of G, there is a connected subgraph L of $G - K$ that meets all elements of $\mathscr{R}$. Upon contracting H and L, it is easy to see that the graph $H \cup L \cup P \cup \rho_0 \cup \rho_h \cup \rho_i \cup \rho$ contains a minor isomorphic to $K_{3,3}$; a contradiction. □

Let $\mathscr{R}$ be a finite non-empty set of rays in a graph G, and let H and K be finite subgraphs of G. We shall write $H <_{\mathscr{R}} K$ if
(R1) both H and K meet all rays in $\mathscr{R}$;
(R2) H and K are disjoint; and
(R3) for every ray $\rho \in \mathscr{R}$ and every $h \in V(H \cap \rho)$ and $k \in V(K \cap \rho)$, we have $h <_\rho k$.

Lemma 2.2. *Let G be a locally finite planar graph with exactly one end. Let $\mathscr{R}$ be a finite non-empty set of pairwise-disjoint rays in G, and let H be a finite subset of G meeting all rays in $\mathscr{R}$. Then there is a path P in G that collates $\mathscr{R}$ and such that $H <_{\mathscr{R}} P$.*

Proof. First we show the following.
(1) For every finite subgraph K of G that meets all rays in $\mathscr{R}$, there is a subgraph K' of G such that $K <_{\mathscr{R}} K'$.

For every ray ρ in $\mathscr{R}$, let $V'(\rho)$ denote the vertices of ρ that are not in the infinite component of $\rho - K$. Let L be the subgraph of G induced by the vertices in $V(K) \cup (\bigcup_{\rho \in \mathscr{R}} V'(\rho))$. It is clear that, since the set $\mathscr{R}$ is finite, the graph L is also finite. Since G has only one end, for every two rays in $\mathscr{R}$, there is a path in $G-L$ that joins these two rays. Thus, as $\mathscr{R}$ has only finitely many elements, we conclude that (1) holds.

We apply (1) with $K=H$ to obtain a finite subgraph H' of G such that $H<_{\mathscr{R}}H'$. We apply (1) again with $K=H'$ to obtain a subgraph H'' such that $H'<_{\mathscr{R}}H''$. Let T' be a minimal connected subgraph of H' that meets all the rays in $\mathscr{R}$. Clearly, T' is a tree. Suppose that T' is not a path. Then T' has at least three vertices v_1, v_2, and v_3 whose degree in T' is one. By the minimality of T', these vertices lie in distinct rays in $\mathscr{R}$, say ρ_1, ρ_2, and ρ_3, respectively. By construction, for each $i\in\{1,2,3\}$, vertex v_i lies on ρ_i between H and H''. It is easy to see that the subgraph of G that is the union of ρ_1, ρ_2, ρ_3, H, T', and H'' contains a minor isomorphic to $K_{3,3}$, contradicting the assumption that G is planar. Thus T' is a path and the conclusion follows from Lemma 2.1. $\square$

The next few lemmas describe the order in which paths and cycles of a graph G meet the members of a finite set $\mathscr{R}$ of pairwise-disjoint rays of G. To formalize the notion of this order, we introduce the following definitions. Two sequences of distinct members of $\mathscr{R}$ are *path-equivalent* if they are equal, or one can be obtained from the other by reversing the order. Two such sequences $(\rho_1,\rho_2,\dots,\rho_m)$ and $(\rho'_1,\rho'_2,\dots,\rho'_{m'})$ are *cycle-equivalent* if $m=m'$ and there are integers a and b such that, for all $i\in\{1,2,\dots,m\}$, $\rho'_i=\rho_{a+bi}$ where $b\in\{-1,1\}$ and all subscripts are read modulo m. Suppose P is a path of G that collates $\mathscr{R}$ and $\rho_1,\rho_2,\dots,\rho_m$ are the rays of $\mathscr{R}$ listed in the order they are met by P. Then the *$\mathscr{R}$-trace* of P, written $\mathrm{tr}_{\mathscr{R}}(P)$, is the path-equivalence class of $(\rho_1,\rho_2,\dots,\rho_m)$. Similarly, suppose C is a cycle that collates $\mathscr{R}$ and $\sigma_1,\sigma_2,\dots,\sigma_m$ are the rays of $\mathscr{R}$ listed in the order they are met by C. Then the *$\mathscr{R}$-trace* of C, written $\mathrm{tr}_{\mathscr{R}}(C)$, is the cycle-equivalence class of $(\sigma_0,\sigma_1,\dots,\sigma_m)$.

Lemma 2.3. *Let G be a locally finite planar graph with exactly one end. Suppose $\mathscr{R}$ is a finite set of pairwise disjoint rays in G, and P_1 and P_2 are paths collating $\mathscr{R}$ such that $P_1<_{\mathscr{R}}P_2$ and $\mathrm{tr}_{\mathscr{R}}(P_1)=\mathrm{tr}_{\mathscr{R}}(P_2)$. Then at least one of the following holds*:
(i) *G has a cycle C collating $\mathscr{R}$ such that $P_1<_{\mathscr{R}}C$; or*
(ii) *G has a path P_3 collating $\mathscr{R}$ such that $P_2<_{\mathscr{R}}P_3$ and $\mathrm{tr}_{\mathscr{R}}(P_2)=\mathrm{tr}_{\mathscr{R}}(P_3)$.*

Proof. Let H be a subgraph of G that is the union of P_2 and all finite components of $\rho-P_2$ over all $\rho\in\mathscr{R}$. Clearly, H is finite. Upon applying Lemma 2.2, we conclude that G has a path P_3 that collates $\mathscr{R}$ and is such that $P_2<_{\mathscr{R}}P_3$. Similarly, G has a path P_4 that collates $\mathscr{R}$ and is such that $P_3<_{\mathscr{R}}P_4$. If $\mathrm{tr}_{\mathscr{R}}(P_2)=\mathrm{tr}_{\mathscr{R}}(P_3)$, then (ii) holds.

We may now assume that $\mathrm{tr}_{\mathscr{R}}(P_2)\neq\mathrm{tr}_{\mathscr{R}}(P_3)$. Let $\mathrm{tr}_{\mathscr{R}}(P_2)$ be the equivalence class of the sequence $(\rho_1,\rho_2,\dots,\rho_n)$, and let π be a permutation of $\{1,2,\dots,n\}$ such that $\mathrm{tr}_{\mathscr{R}}(P_3)$ is the equivalence class of $(\rho_{\pi(1)},\rho_{\pi(2)},\dots,\rho_{\pi(n)})$. Since the traces of P_2 and P_3 differ, there is a number k in $\{1,2,\dots,n-1\}$ such that $|\pi(k+1)-\pi(k)|>1$. By symmetry, we may assume that $\pi(k+1)>\pi(k)$. If $\pi(k+1)-\pi(k)=n-1$, then it is easy to see that (i) holds. Thus, we may assume that $\pi(k+1)-\pi(k)<n-1$, which

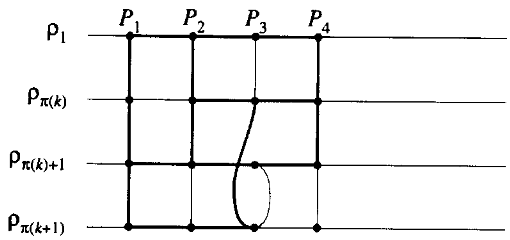

Fig. 2.

implies that $\pi(k+1) \neq n$ or $\pi(k) \neq 1$. By symmetry, we may assume that the latter holds. Then it follows that G has a minor isomorphic to $K_{3,3}$, as illustrated in Fig. 2; a contradiction. □

Lemma 2.4. *Let G be a locally finite planar graph with exactly one end, and let $\mathcal{R}$ be a finite set of pairwise disjoint rays. Suppose C_1 and C_2 are two cycles of G such that $C_1 <_{\mathcal{R}} C_2$ and each collates $\mathcal{R}$. Then* $\mathrm{tr}_{\mathcal{R}}(C_1) = \mathrm{tr}_{\mathcal{R}}(C_2)$.

Proof. Suppose the lemma fails. Let $(\rho_1, \rho_2, \ldots, \rho_n)$ be a sequence in the equivalence class $\mathrm{tr}_{\mathcal{R}}(C_1)$. Let π be a permutation of $\{1, 2, \ldots, n\}$ such that $\mathrm{tr}_{\mathcal{R}}(C_2)$ is the cycle-equivalence class of $(\rho_{\pi(1)}, \rho_{\pi(2)}, \ldots, \rho_{\pi(n)})$. Since the traces of C_1 and C_2 differ, there is a number k in $\{1, 2, \ldots, n-1\}$ such that $1 < |\pi(k+1) - \pi(k)| < n-1$. Without loss of generality, we may assume that $\pi(k) < \pi(k+1)$. Then $\pi(k) \neq 1$ or $\pi(k+1) \neq n$. By symmetry, we may assume the former. Then the graph $C_1 \cup C_2 \cup \rho_1 \cup \rho_{\pi(k)} \cup \rho_{\pi(k)+1} \cup \rho_{\pi(k+1)}$ contains a minor isomorphic to $K_{3,3}$; a contradiction. □

Lemma 2.5. *Let G be a planar locally finite graph and let $\mathcal{R}$ be a finite set of pairwise disjoint rays of G that are in the same end of G. Let H be a finite connected subgraph of G that meets all members of $\mathcal{R}$, and let C be a cycle in $G - H$ that collates at least three members of $\mathcal{R}$. Then C meets every ray in $\mathcal{R}$.*

Proof. Let $\{\rho_1, \rho_2, \rho_3\}$ be a 3-element subset of $\mathcal{R}$ that is collated by C. Suppose that ρ is a ray in $\mathcal{R}$ that avoids C. Let $K = H \cup C \cup [H, C]_{\rho_1} \cup [H, C]_{\rho_2} \cup [H, C]_{\rho_3}$. Then, as all rays in $\mathcal{R}$ are in the same end of G, there is a connected subgraph L in $G - K$ that meets all of ρ, ρ_1, ρ_2, and ρ_3. Upon contracting all edges in $H \cup L$, it becomes clear that the graph $H \cup L \cup C \cup [H, C]_{\rho_1} \cup [H, C]_{\rho_2} \cup [H, C]_{\rho_3}$ has a minor isomorphic to K_5; a contradiction. □

3. Grids and cylinders

Let n be a positive integer. The *$(n \times n)$-grid* is the subgraph of $G_{\mathbb{Z} \times \mathbb{Z}}$ induced by the subset $\{0, 1, \ldots, n-1\} \times \{0, 1, \ldots, n-1\}$ of its vertex set. The *$(n \times n)$-cylinder* is the graph obtained from the $n \times n$-grid by adding edges joining vertices $(i,0)$ to $(i,n-1)$, for all i in $\{0,1,\dots,n-1\}$. It turns out in our studies, however, that finite grids and cylinders are easier to handle if their vertices are pairs of binary sequences, rather than pairs of integers. Thus, we shall denote by Γ_n the graph with vertex set $[2^n] \times [2^n]$ that is isomorphic to the $(2^n \times 2^n)$-grid, so that the vertex (i,j) of the grid corresponds to the vertex (α_i, α_j) of Γ_n, where α_i and α_j are the binary expansions of i and j with the appropriate numbers of leading zeros. Using the same correspondence as above, we define Θ_n to be the graph isomorphic to the $(2^n \times 2^n)$-cylinder.

The following well-known theorem of Robertson et al. [5] states that every finite planar graph is a minor of a sufficiently large grid.

Theorem 3.1. *For every planar graph G, there is a positive integer N such that G is isomorphic to a minor Γ_N.*

The remainder of this section contains two results, which are technical modifications of Theorem 3.1, and which will be useful for proving the main theorems of this paper. The first of these modifications strengthens Theorem 3.1 by specifying that the isomorphism in Theorem 3.1 can be chosen so that some parts of the boundary of the infinite face of G are mapped to the appropriate parts of the boundary of the grid. The second of these modifications pertains to cylinders, rather than grids. Before formally stating these results, we need some preparation.

Let m be a positive integer and let $A_m = \{(\mathbf{0}_m, \beta): \beta \in [2^m]\}$, and $B_m = \{(\mathbf{1}_m, \beta): \beta \in [2^m]\}$. Let Γ_m^0 and Γ_m^1 denote the subgraphs of Γ_m that are induced by the sets A_m and B_m, respectively. Similarly, let Θ_m^0 and Θ_m^1 denote the subgraphs of Θ_m that are induced by A_m and B_m. Suppose that G is a plane graph and φ is one of its faces. A *flat m-attachment* of G is a pair (P,h) where P is the null graph or a path in the boundary of φ, and h is an isomorphism from P to a minor of Γ_m^0. Suppose now that C is the null graph or the cycle that forms the boundary of φ, and k is an isomorphism from C to Θ_m^0. The pair (C,k) will be called a *round m-attachment* of G.

Let (H,h) be either a flat m-attachment of G or a round m-attachment of G, and let n be an integer with $n \geqslant m$. Suppose that k is an isomorphism from H to, respectively, Γ_n^0 or Θ_n^0. Then k *agrees* with h if, for every vertex v of H, there are vertices $(\mathbf{0}_m, \alpha) \in h(v)$ and $(\mathbf{0}_n, \beta) \in k(v)$ such that α is an initial segment of β.

A *flat n-segment* is a triple $(G,(P,h),Q)$ consisting of a graph G, a flat n-attachment (P,h), and a path Q that lies in the boundary of φ and is disjoint from P. A *round n-segment* is a triple $(G,(C,h),D)$ consisting of the graph G, a round n-attachment (C,h), and a cycle D that bounds a face of G and is disjoint from C. Let $(G,(H,h),K)$ be a flat n-segment or a round n-segment. An isomorphism f from G to a minor of Γ_n or to a minor of Θ_n *agrees* with $(G,(H,h),K)$ if f restricted to P agrees with h, and, for every vertex v of K, the set $V(f(v))$ contains an element $(\mathbf{1}_n, \beta)$ for some $\beta \in [2^n]$.

The two modifications of Theorem 3.1 referred to earlier in this section are stated below.

Theorem 3.2. *For every positive integer n and every flat n-segment $(G,(P,h),Q)$, there is an integer N exceeding n and an isomorphism f from G to a minor of Γ_N that agrees with $(G,(P,h),Q)$.*

Theorem 3.3. *For every positive integer and every round n-segment $(G,(C,k),D)$, there is an integer N exceeding n and an isomorphism from G to a minor of Θ_N that agrees with $(G,(C,k),D)$.*

The remainder of this section will be devoted to proving Theorems 3.2 and 3.3. Before presenting formal proofs, however, we need some more terminology and some auxiliary results.

For a binary sequence α, let $\alpha^{\#}$ denote the number whose binary expansion consists of a zero followed by the point, followed by the elements of α. Observe that the graph Γ_n is planar with the obvious plane embedding that maps every vertex (α,β) of Γ_n to the point $(\alpha^{\#},\beta^{\#})$ of the plane with the edges of Γ_n mapping to the appropriate line segments. In what follows, we shall identify the graph Γ_n with the plane graph induced by this embedding, and we shall, for instance, refer to the faces of Γ_n.

Suppose C is a cycle in Γ_n. Then C induces a decomposition of Γ_n as $C\cup C^{\text{in}}\cup C^{\text{out}}$, where C^{in} and C^{out} lie, respectively, inside and outside C in the above plane embedding of Γ_n, and are such that $C^{\text{in}}\cap C^{\text{out}}$ is edgeless and contained in C.

Suppose now that J is a minor of Γ_n that is isomorphic to a cycle. An *innermost cycle* of J is a cycle M in the expansion $\Sigma(J)$ satisfying the following conditions.

(i) The intersection of every edge of J with M is a path.

(ii) If L is a cycle in $\Sigma(J)$ whose intersection with every edge of J is a path, then $M^{\text{in}}\subseteq L^{\text{in}}$.

It is clear that every minor J of Γ_n that is isomorphic to a cycle contains exactly one innermost cycle. The concept of the innermost cycle is illustrated in Fig. 3.

If a binary sequence α contains at least one zero, then let $\alpha^* = \{\alpha+00,\alpha+01,\alpha+10\}$; and if α consists of all ones, then let $\alpha^* = \{\alpha+00,\alpha+01,\alpha+10,\alpha+11\}$.

Suppose now that n is an integer exceeding one and (α,β) is a vertex of Γ_n. Let $v_{\alpha\beta}$ be the subgraph of Γ_{n+2} that is induced by the vertices in $\alpha^*\times\beta^*$. If, additionally, $\alpha\neq\mathbf{1}$, then let $e_{\alpha\beta}$ be the path in Γ_{n+2} that is induced by the vertices in $\{(\alpha+10,\beta+01),(\alpha+11,\beta+01),(\alpha^{+}+00,\beta+01)\}$. Similarly, if $\beta\neq\mathbf{1}$, then let $f_{\alpha\beta}$ be the path in Γ_{n+2} that is induced by the vertices in $\{(\alpha+01,\beta+10),(\alpha+01,\beta+11),(\alpha+01,\beta^{+}+00)\}$.

The minor of Γ_{n+2} whose vertex set is $\{v_{\alpha\beta}\colon (\alpha,\beta)\in V(\Gamma_n)\}$ and whose edge set is $\{e_{\alpha\beta}\colon (\alpha,\beta)\in V(\Gamma_n),\ \alpha\neq\mathbf{1}\}\cup\{f_{\alpha\beta}\colon (\alpha,\beta)\in V(\Gamma_n),\ \beta\neq\mathbf{1}\}$ will be denoted by Γ_n'. Let ι_n denote the natural isomorphism from Γ_n to Γ_n' that maps each (α,β) to $v_{\alpha\beta}$. Let ι denote the function defined on $\bigcup_{n\in\mathbb{N}}\Gamma_n$ whose restriction to each Γ_n is ι_n. It is clear that if J is a minor of Γ_n, then the function ι determines in a natural way a minor ιJ of Γ_{n+2}, which is isomorphic to J. More specifically, a vertex W of J, which is a subgraph of Γ_n, corresponds to $(\bigcup_{v\in V(W)}\iota(v))\cup(\bigcup_{e\in E(W)}\iota(e))$, and, similarly, an edge

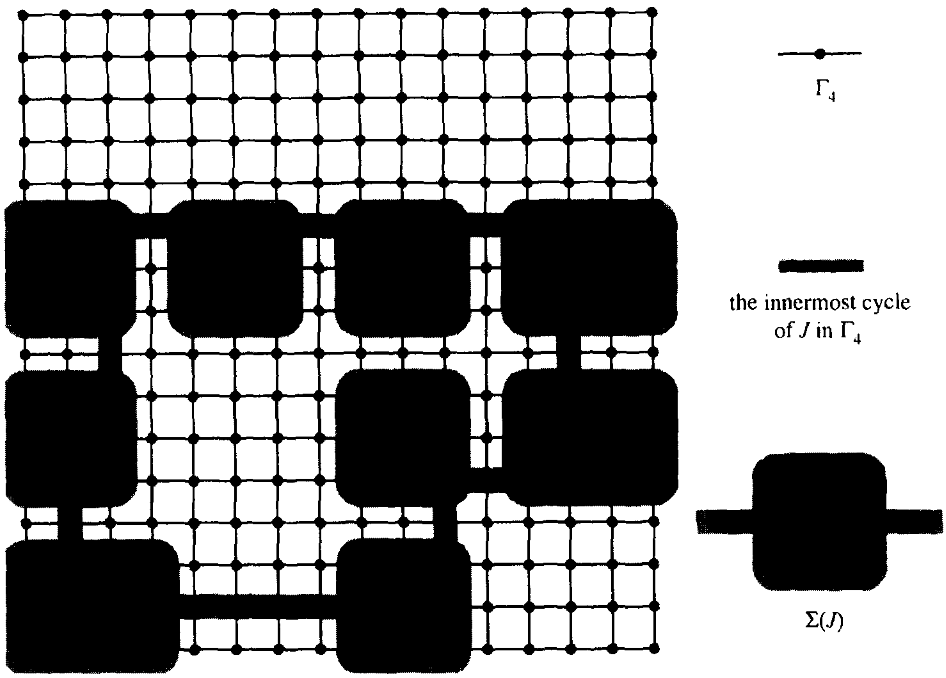

Fig. 3.

F of J corresponds to $(\bigcup_{v\in V(F)} \imath(v))\cup(\bigcup_{e\in E(F)} \imath(e))$. Also, if f is an isomorphism from a graph G to a minor J of Γ_n, then the composition $\imath f$ is an isomorphism from G to the minor $\imath J$ of Γ_{n+2}. For an illustration of Γ_2' and $\imath_2$ see Fig. 4.

The following lemmas describe the properties of $\imath$ that will be used later in this section. The proofs of these lemmas are routine, if sometimes tedious, and so they are left for the reader.

Lemma 3.6. *Suppose (H,h) is either a flat n-attachment or a round n-attachment of a graph G. Then the composition $\imath h$ agrees with h.*

Lemma 3.7. *Suppose $S=(G,(P,h),Q)$ is either a flat n-segment or a round n-segment and f is an isomorphism from G to a minor of, respectively, Γ_n or Θ_n that agrees with S. Then the composition $\imath f$ also agrees with S.*

Lemma 3.8. *If P is a path in Γ_n joining two of its subgraphs H and K, then $\imath(P)$ contains a path P' that joins $\imath(H)$ to $\imath(K)$ and has more vertices than P.*

Lemma 3.9. *Suppose G is a subgraph of Γ_n and C is a cycle of G that bounds a finite face. Then the innermost cycle of $\imath(C)$ also bounds a finite face of $\Sigma(\imath(G))$.*

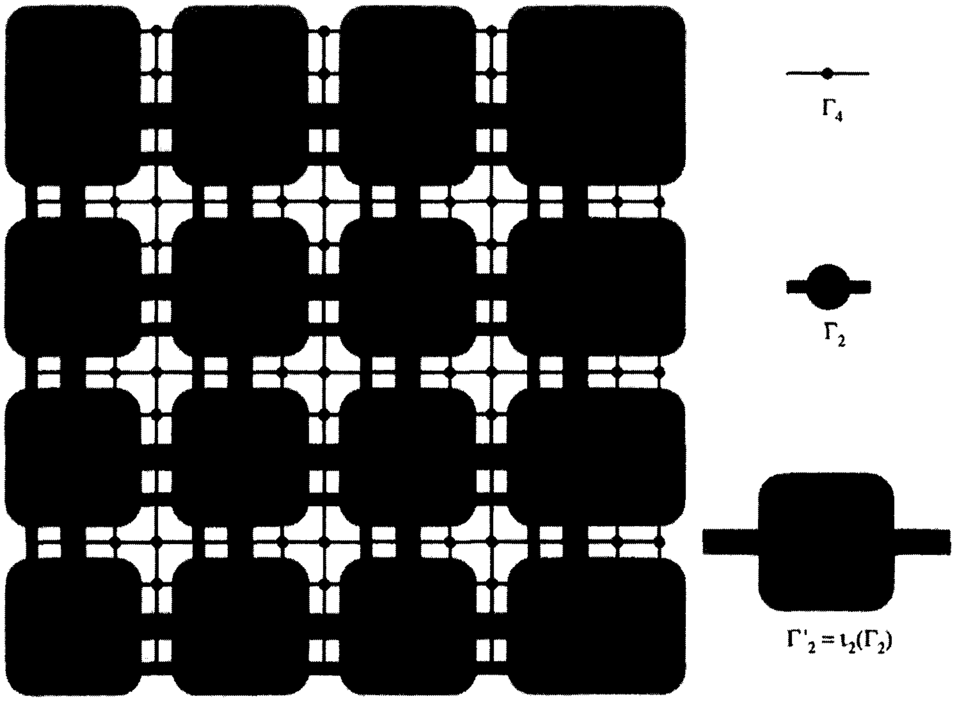

Fig. 4.

Lemma 3.10. *Suppose f is an isomorphism from a cycle C to a minor J of Γ_n, and u and v are distinct vertices of C. Let D denote the innermost cycle of J. Then there is a path in D^{in} that is internally disjoint from $\Sigma(\iota J)$, and joins $\iota(f(u))$ to $\iota(f(v))$.*

Now we are ready to prove Theorem 3.2.

Proof of Theorem 3.2. First, we shall argue that G may be assumed to be non-separable. Suppose that G has more than one block and construct a plane graph G_3 by following steps (1)–(3) below.

(1) If no block of G contains both P and Q, then construct G_1 from G by adding two edges e and f so that the paths P and Q together with the edges e and f form a cycle in the boundary of the infinite face φ_1 of G_1. If both P and Q are contained in the same block, then let $G_1 = G$, and $\varphi_1 = \varphi$.

(2) If the boundary of φ_1 contains edges from more than one block of G_1, then redraw the blocks of G_1 other than the one containing P in the faces other than φ_1. Call the resulting graph G_2 and its infinite face φ_2.

(3) Form G_3 from G_2 by adding edges, if necessary, so that G_3 is 2-connected and plane.

It is easy to verify that $(G_3,(P,h),Q)$ is also a flat n-segment and that an isomorphism from G_3 to Γ_N that agrees with $(G_3,(P,h),Q)$ induces an isomorphism from G to Γ_n that agrees with $(G,(P,h),Q)$. Hence G may be assumed to be non-separable.

Let C denote the cycle of G that bounds the infinite face and let p denote the number of elements in $E(G)-E(C)$. The following is an easy consequence of a well-known property of non-seperable graphs (see [6]).

(4) *There is a sequence of plane non-separable graphs $G_0, G_1, \dots, G_p$ such that $G_0 = C, G_p = G$, and, for each $i \in \{1,2,\dots,p\}$, the graph G_i has been obtained from G_{i-1} by adding a path that meets G_{i-1} only in its endvertices.*

The proof will proceed by induction on p. We shall strengthen the statement of (3.2) by requiring additionally that (5) and (6) below hold. Condition (5) will facilitate the induction; and condition (6), while not needed to prove Theorem 3.2, will be used in proving Theorem 3.3.

(5) *For every cycle K of G that bounds a finite face of G, the innermost cycle of $f(K)$ in Γ_N bounds a finite face of $\Sigma(f(G))$.*

Let $t'_0, t'_1, \dots, t'_{q'}$ be the vertices of T' listed in the order they appear on T' where $t'_0 = V(T) \cap V(P)$. Similarly, let $t''_0, t''_1, \dots, t''_{q''}$ be the vertices of T'' listed in the order they appear on T'' where $t''_0 = V(T'') \cap V(P)$.

(6) *If $q' = q''$, then, for every i in $\{0,1,\dots,q'\}$, there is an $\alpha \in [2^N]$ for which $(\alpha, \mathbf{0}) \in V(f(t'_i))$ and $(\alpha, \mathbf{1}) \in V(f(t''_i))$.*

Suppose first that $p=0$. Then $G=C$ and so G can be expressed as a union of four pairwise edge-disjoint paths P, Q, T', and T''. Let N be the smallest integer such that $2^N \geqslant \max\{2^n, |V(Q)|, |V(T)|, |V(T'')|\}$. Clearly, there is an isomorphism f from C to a minor of the boundary of the infinite face of Γ_{n_0} that agrees with $(C,(P,h),Q)$ and satisfies (6). It is also clear that (5) holds for the graph C and the isomorphism f. Thus the claim holds if $p=0$.

Now suppose that p is a positive integer and the theorem holds for all smaller values of p. By the inductive hypothesis, there is an integer M and an isomorphism g from G_{p-1} to a minor of Γ_M that agrees with $(G_{p-1},(P,h),Q)$. Let R be a path in G_p that meets G_{p-1} only in its endvertices u and v, and is such that $G_p = G_{p-1} \cup R$. As G_p is a plane graph and R meets G_{p-1} only in its endvertices, there is a finite face ψ of G_{p-1} such that R splits ψ into ψ_1 and ψ_2 which form faces of G_p. Let D denote the cycle in G_{p-1} that is the boundary of ψ, and let E denote the innermost cycle of $g(D)$ in Γ_{p-1}. Then, by applying (5) to the graph G_{p-1} and the isomorphism g, we conclude that E bounds a finite face ψ' of $\Sigma(g(G_{p-1}))$. It is clear that each of $g(u)$ and $g(v)$ meets E. Thus, upon combining Lemma 3.10 with a multiple application of Lemma 3.8, we conclude that there is a positive integer k such that $(\iota^k g(D))^{\text{in}}$ contains a path R' that joins $\iota^k g(u)$ to $\iota^k g(v)$ and has at least as many vertices as R. Now let $N = M + 2k$. It follows from Lemma 3.7 that $\iota^k g$ is an isomorphism from G_{p-1} to Γ_N that agrees with $(G_{p-1},(P,h),Q)$. It is clear that $\iota^k g$ can be extended to an isomorphism f from G_p to Γ_N that maps R to a minor of R' and that also agrees with $(G_{p-1},(P,h),Q)$. It is clear that (5) holds for G_p and f. Moreover, it follows from the inductive hypothesis and from the definition of ι that f satisfies (6). $\square$

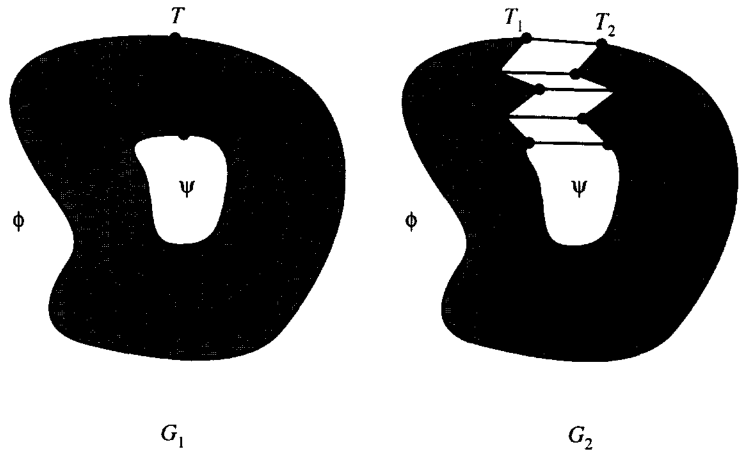

Fig. 5.

We conclude this section by using Theorem 3.2 to prove Theorem 3.3.

Proof of Theorem 3.3. In the first part of the proof, we shall construct a flat n-segment $(H,(P,h),Q)$ from $(G,(C,k),D)$, and then we shall use Theorem 3.2 to derive Theorem 3.3.

Let φ and ψ denote the faces of G whose boundaries are, respectively, C and D. Let α_0 denote the lexicographically smallest element α of $[2^n]$ for which $(\mathbf{0}_n, \alpha)$ is a vertex of $k(v_0)$ for some vertex v_0 of C. We construct G_1 from G by adding edges, if necessary, in faces other than φ and ψ so that G_1 has a path T joining v_0 to D. Let $t_0, t_1, \ldots, t_s$ denote the vertices of T, listed here in the order they appear on T and so that $t_0 \in V(C)$ and $t_s \in V(D)$. Let T' and T'' be two copies of T that are disjoint from each other and from G. For each $i \in \{0,1,\ldots,s\}$, let t_i' and t_i'' be the vertices of T' and T'', respectively, that correspond to t_i. A *T-ladder* is a plane graph obtained from the union $T' \cup T''$ by adding edges of the form $\{t_i', t_i''\}$, called the *rungs*, for all $i \in \{0,1,\ldots,s\}$. We construct a plane graph G_2 from G_1 by replacing T in G_1 by a T-ladder, as illustrated in Fig. 5.

Let φ_2 and ψ_2 denote the faces of G_2 that correspond to, respectively, the faces φ and ψ of G, and let C_2 and D_2 be the boundaries of φ_2 and ψ_2. It is easy to see that the isomorphism k from C to a minor of Θ^0_{n+2} can be extended to k_2, which is an isomorphism from C_2 to a minor of Θ^0_{n+2} that agrees with k and is such that $V(k_2(t_0')) \ni (\mathbf{0}_{n+2}, \mathbf{0}_{n+2})$ and $V(k_2(t_1')) \ni (\mathbf{0}_{n+2}, \mathbf{1}_{n+2})$. Let G_3 be the graph obtained from G_2 by removing the rungs of the T-ladder, and let C_3 and D_3 denote the resulting subgraphs of C_2 and D_2, respectively. Let k_3 be the restriction of k_2 to C_3. Then $(G_3,(C_3,k_3),D_3)$ is a flat attachment. Moreover, the boundary of the infinite face of G_3 consists of four edge-disjoint paths C_3, D_3, T', and T'' that satisfy $|V(T')| = |V(T'')|$. Thus, by Theorem 3.2 and (6) in its proof, there is a number N and an isomorphism f_3 from G_3 to a minor of Γ_N that agrees with $(G_3,(C_3,k_3),D_3)$ such that, for each vertex

t_i of T, there is an $\alpha_i \in [2^N]$ so that $(\alpha_i, \mathbf{0}) \in V(f_3(t_i'))$ and $(\alpha_i, \mathbf{1}) \in V(f_3(t_i''))$. Extend f_3 to the isomorphism f_2 from G_2 to a minor of Θ_N by mapping the rung $\{t_i', t_i''\}$ of the T-ladder to the edge $\{(\alpha_i, \mathbf{0}), (\alpha_i, \mathbf{1})\}$. It is clear that f_2 induces an isomorphism f, as desired. □

4. Flat graphs

The main result of this section is the following:

Theorem 4.1. *Every flat graph is minor-equivalent to $G_{\mathbb{Z}\times\mathbb{N}}$.*

Let $G'_{\mathbb{Z}\times\mathbb{N}}$ be the graph that is formed as follows. First take the disjoint union of all of the graphs Γ_n for $n \geqslant 1$, then add new edges that join the vertex $(\mathbf{1}_n, \beta)$ to both $(\mathbf{0}_{n+1}, \beta + 0)$ and $(\mathbf{0}_{n+1}, \beta + 1)$ for all positive integers n and all $\beta \in [2^n]$. It is clear that Theorem 4.1 follows from the following three lemmas.

Lemma 4.2. *If G is flat, then $G_{\mathbb{Z}\times\mathbb{N}}$ is isomorphic to a minor of G.*

Lemma 4.3. *If G is flat, then G is isomorphic to a minor of $G'_{\mathbb{Z}\times\mathbb{N}}$.*

Lemma 4.4. *The graph $G'_{\mathbb{Z}\times\mathbb{N}}$ is isomorphic to a minor of $G_{\mathbb{Z}\times\mathbb{N}}$.*

Observe that upon combining Lemmas 4.3 and 4.4, we conclude that every flat graph is isomorphic to a minor of $G_{\mathbb{Z}\times\mathbb{N}}$. A direct proof of this fact, however, is very messy. Introducing the graph $G'_{\mathbb{Z}\times\mathbb{N}}$ and dealing with Lemmas 4.3 and 4.4 instead, makes for a much cleaner argument. The proof of Lemma 4.4 is straightforward but somewhat tedious to write out in detail; we leave a formal proof to the reader. The remainder of this section will be concerned with proving Lemmas 4.2 and 4.3. Lemma 4.2 will be derived from a result of [4], but before this result can be stated, we need some preparation.

Let G be a graph and let Θ be the pair $(T, (X_t)_{t\in V(T)})$ which consists of a tree T and a multiset whose elements X_t, indexed by the vertices of T, are finite subsets of $V(G)$. For a vertex v of G, we denote by T_v the subgraph of T induced by those vertices t of T for which X_t contains v. For a subgraph H of G, we let $T_H = \bigcup_{v\in V(H)} T_v$. Then Θ is called a *finite tree-decomposition* of G if it satisfies conditions (T1)–(T3) below.

(T1) The union, over all vertices t of T, of the subgraphs of G induced by X_t equals G.

(T2) For every vertex v of G, the subgraph T_v of T is a tree.

(T3) For every ray ρ of T, there is an integer n_ρ such that if t' and t'' are two adjacent vertices of ρ, then $|X_{t'} \cap X_{t''}| \leqslant n_\rho$.

The following lemma is an easy consequence of (T2) and (T3) — we omit the proof.

Lemma 4.5. *If $(T, (X_t)_{t\in V(T)})$ is a finite tree-decomposition of a graph G, and H is a connected subgraph of G, then T_H is a tree.*

One of the results of [4] is the following:

Theorem 4.6. *A graph G has a minor isomorphic to $G_{\mathbb{Z}\times\mathbb{N}}$ if and only if there is no finite tree-decomposition of G.*

We deduce from Theorem 4.6 that Lemma 4.2 is a consequence of the following:

Lemma 4.7. *No graph with a thick end has a finite tree-decomposition.*

Proof. Suppose the lemma fails and G is a counterexample. Let ε be a thick end of G, and let $(T,(X_t)_{t\in V(T)})$ be a finite tree-decomposition of G. Let ρ be a ray from ε. From Lemma 4.5 and the fact that each set X_t is finite, we conclude that, for every vertex t of T_ρ, all but finitely many vertices of ρ are in the same connected component of the forest obtained from T by deleting t. As ρ is infinite, this immediately implies that

(1) *the graph T_ρ has exactly one ray.*

For a ray ρ of G, let ρ' denote the unique ray of T_ρ. Suppose now that ρ and σ are two distinct rays in ε. We shall show that

(2) *ρ' and σ' are in the same end of T.*

Suppose not. Then there is a vertex t of T such that the infinite connected components of $\rho' - \{t\}$ and $\sigma' - \{t\}$ are in distinct connected components of $T - \{t\}$. Since ρ and σ are both in ε, there are infinitely many pairwise-disjoint paths in G each of which joins the infinite connected component of $\rho - \{t\}$ to the infinite connected component of $\sigma - \{t\}$. By Lemma 4.5, for every such path P, the graph T_P is connected, and hence its vertex set contains t. Thus, every such path has at least one of its vertices in X_t; a contradiction to the finiteness of X_t. This proves (2).

Now it follows from (2) that there is a ray τ of T such that, for every ray ρ from ε, the unique ray ρ' of T_ρ is equivalent to τ. From (T3), there is a number n_τ such that if t' and t'' are two adjacent vertices of τ, then $|X_{t'} \cap X_{t''}| \leqslant n_\tau$. Since ε is thick, there a finite set $\mathscr{R}$ of pairwise-disjoint rays from ε that has more than n_τ elements. Since $\mathscr{R}$ is finite, and, for each of its elements ρ, the ray ρ' is equivalent to τ, there is a vertex s of τ that meets ρ' for all $\rho \in \mathscr{R}$. But this is impossible, since the rays in $\mathscr{R}$ are pairwise disjoint and the cardinality of $\mathscr{R}$ exceeds the cardinality of X_s. □

To conclude this section, it remains to show Lemma 4.3.

Proof of Lemma 4.3. For a finite subgraph A of G, let $\bar{A}$ denote the subgraph of G that is the union of A and all finite bridges of A in G. Clearly, as A is finite and G is locally finite, $\bar{A}$ is also finite.

Without loss of generality, we may assume that G is a plane graph. Since G is flat, it has a finite subgraph H and a set $\mathscr{R}$ of pairwise-disjoint rays such that $|\mathscr{R}| \geqslant 3$ and no cycle in $G - H$ collates $\mathscr{R}$. We shall inductively define an ascending sequence $A_0, A_1, \ldots$ of finite connected subgraphs of G as follows. Since each of H and $\mathscr{R}$ is finite, there is a finite connected subgraph A_0 of G that contains H and meets all rays in $\mathscr{R}$. Suppose i is a positive integer and A_{i-1} has been defined so that it is connected. Clearly, $\overline{A_{i-1}}$ is finite and connected. Let A_i be the subgraph of G induced by the vertices whose distance to $\overline{A_{i-1}}$ is less than two. Then A_i is also finite and connected.

Let $B_0 = A_0$. For a positive integer i, let $B_i = A_i \backslash \overline{A_{i-1}}$, and let V_i denote the vertices common to B_{i-1} and B_i. From the construction, we conclude the following:

(1) *If i and j are non-negative integers such that $|j-i| \geqslant 2$, then B_i and B_j are disjoint.*

(2) *If i and j are non-negative integers such that $|j-i| = 1$, then $B_i \cap B_j$ is a non-null edgeless graph with vertex set V_i.*

(3) *B_0 has exactly one bridge $\bigcup_{j>0} B_j$, and the set of the vertices of attachment of this bridge is V_1.*

(4) *For every positive integer i, the graph B_i has two bridges: the finite $\overline{A_{i-1}}$, and the infinite $\bigcup_{j>1} B_j$. The sets of vertices of attachment of these bridges are V_i and V_{i+1}, respectively.*

It follows from (3) that all vertices in V_1 lie in the same face φ of B_0. Hence, by adding edges to B_0 if necessary, we may form a plane graph B_0' that has a path P_1 in the boundary of the infinite face of B_0 such that $V(P_1) \supseteq V_1$.

Let now i be a positive integer. From (4) we conclude that there is a face φ of B_i whose boundary contains all elements of V_i. Likewise, there is a face φ' of B_i whose boundary contains all elements of V_{i+1}. We shall show that

(5) $\varphi = \varphi'$.

Suppose not. Then B_i has a cycle C that meets every path joining a vertex in V_i to a vertex in V_{i+1}. Observe that, for each ray ρ in $\mathscr{R}$, the intersection $\rho \cap B_i$ is a path joining a vertex of V_i to a vertex in V_{i+1}, and hence meeting C. Let D be a subpath of C that has its endvertices in different rays of $\mathscr{R}$ and is internally disjoint from $\bigcup_{\rho \in \mathscr{R}} \rho$. Then the path $P = C \backslash D$ is contained in $G \backslash H$ and meets all rays in $\mathscr{R}$. By Lemma 2.1, there is a path Q that is contained in $P \cup (\bigcup_{\rho \in \mathscr{R}} P_\rho)$ and collates $\mathscr{R}$. Let ρ and σ denote the elements of $\mathscr{R}$ that contain the endvertices of Q. It follows that the graph $Q \cup (C \backslash D) \cup \rho \cup \sigma$ contains a cycle that avoids H and collates $\mathscr{R}$; a contradiction. Hence (5) follows.

Let $v_1, v_2, \ldots, v_n$ be the list of vertices from $V_i \cup V_{i+1}$ (with possible repetitions) in the cyclic order in which they appear on the boundary of φ. By (3), B_i has two bridges $\overline{A_{i-1}}$ and $\bigcup_{j>i} B_j$, whose sets of vertices of attachments are V_i and V_{i+1}. Thus, as G is a plane graph and each of its bridges is connected, it follows that there are integers a and b such that $V_i = \{v_a, v_{a+1}, \ldots, v_{a+b}\}$ where the arithmetic is carried out modulo n. Hence, by adding edges to B_i if necessary, we form a plane graph B_i' that has two disjoint paths P_i and P_{i+1} in the boundary of its infinite face such that $V(P_i) \supseteq V_i$ and $V(P_{i+1}) \supseteq V_{i+1}$.

We shall use the graphs B_i' to inductively define a sequence of flat segments as follows. Let $h_\emptyset$ denote the trivial isomorphism between null graphs. Then $(B_0', (\emptyset, h_\emptyset), P_1)$

is a flat segment. Hence, by Theorem 3.2, there is an integer n_0 and an isomorphism f_0 from B'_0 to a minor of Γ_{n_0} that agrees with $(B'_0,(\emptyset,h_\emptyset),P_1)$. Inductively, suppose that i is a positive integer, $n_0,n_1,\ldots,n_{i-1}$ is an increasing sequence of integers, $(B'_{i-1},(P_{i-1},h_{i-1}),P_i)$ is a flat segment, and f_{i-1} is an isomorphism from B'_{i-1} to a minor of Γ_{n_i-1} that agrees with $(B'_{i-1},(P_{i-1},h_{i-1}),P_i)$.

Suppose v is a vertex of P_i. Then the set $V(f_{i-1}(v))$ contains an element $(\mathbf{1},\beta_v)$ for some $\beta_v\in[2^{n_{i-1}}]$. Define the isomorphism h_i from P_i to a minor of $\Gamma^0_{n_{i-1}}$ by letting $h_i(v)=(\mathbf{0},\beta_v)$. Then $(B'_i,(P_i,h_i),P_{i+1})$ is a flat segment, and by Theorem 3.2, there is an integer n_i exceeding n_{i-1} and an isomorphism from B'_i to a minor of Γ_{n_i} that agrees with $(B'_i,(P_i,h_i),P_{i+1})$.

Suppose α and β are binary sequences of positive length such that α can be obtained from β by deleting the last entry. Let i denote the length of α, and let $P_{\alpha\beta}$ be the path in $G'_{\mathbb{Z}\times\mathbb{N}}$ that is induced by the vertices in the set $\{(\alpha,\gamma)\colon \gamma\in[2^i]\}\cup\{(\beta,\mathbf{0}_{i+1})\}$. Now let j be an integer exceeding i. Suppose that $\alpha\in[2^i]$ and $\beta\in[2^j]$ are such that α is an initial segment of β. Let $\alpha=\alpha_i,\alpha_{i+1},\ldots,\alpha_j=\beta$ be the sequence of elements of $[2^{<\omega}]$ such that each α_m is otbained from α_{m+1} by deleting the last entry. Then we let $P_{\alpha\beta}$ be the path in $G'_{\mathbb{Z}\times\mathbb{N}}$ that is the union of the paths $P_{\alpha_m\alpha_{m+1}}$ over all m in $\{i,i+1,\ldots,j-1\}$.

Finally, we shall combine all isomorphisms f_i into an isomorphism f from G to a minor of $G'_{\mathbb{Z}\times\mathbb{N}}$. For a vertex v of G, we define $f(v)$ as follows. Suppose first that $v\notin\bigcup_{i\geqslant 1}V_i$. Then it follows from (1)–(3) that there is exactly one integer i such that $v\in V(B_i)$. We let $f(v)=f_i(v)$. Suppose now that $v\in V_i$ for some positive integer i. Then $v\in V(P_{i-1})$, and thus $V(f_{i-1}(v))$ contains an element of the form $(\mathbf{1},\alpha)$ for some $\alpha\in[2^{n_{i-1}}]$. Then $h_i(v)=(\mathbf{0},\alpha)$, and, as f_i agrees with $(B'_i,(P_i,h_i),P_{i+1})$, the set $V(f_i(v))$ contains a vertex of the form $(\mathbf{0},\beta)$ for some binary sequence $\beta\in[2^{n_i}]$ that has α as the initial segment. We let $f(v)=f_{i-1}(v)\cup P_{\alpha\beta}\cup f_i(v)$.

To define f on the edge set of G, observe from (1)–(3) that every edge e of G is contained in exactly one B'_i. We let $f(e)=f_i(e)$. It is easy to verify that f satisfies the conclusion of Lemma 4.3. □

5. Round graphs

In this section, we prove analogs of the results from Section 4 for round graphs. The main result of this section is the following:

Theorem 5.1. *Every round graph is minor-equivalent to* $G_{\mathbb{Z}\times\mathbb{Z}}$.

Just as in Section 4, we shall introduce a new graph that is the most convenient representative of the minor-equivalence class of round graphs. Let $G'_{\mathbb{Z}\times\mathbb{Z}}$ be the graph obtained by taking the disjoint union of the graphs Θ_n over all integers $n\geqslant 2$, and then, for each $n\geqslant 2$ and each $\beta\in[2^n]$, adding two new edges that join the vertex $(\mathbf{1}_n,\beta)$ to $(\mathbf{0}_{n+1},\beta+0)$ and to $(\mathbf{0}_{n+1},\beta+1)$. As noted in Proposition 1.10, $G_{\mathbb{Z}\times\mathbb{Z}}$ is round. Thus, to prove Theorem 5.1, it suffices to show that all round graphs are in the same minor-equivalence class. This will be accomplished by proving the following two lemmas.

Lemma 5.2. *If G is round, then $G'_{\mathbb{Z}\times\mathbb{Z}}$ is isomorphic to a minor of G.*

Lemma 5.3. *If G is round, then G is isomorphic to a minor of $G'_{\mathbb{Z}\times\mathbb{Z}}$.*

The remainder of this section is devoted to proving Lemmas 5.2 and 5.3.

Proof of Lemma 5.2. Observe that, roughly speaking, the graph $G'_{\mathbb{Z}\times\mathbb{Z}}$ is the union of an infinite set of pairwise-disjoint cycles and a subdivision of the infinite binary tree. This observation captures the main idea of the proof.

It follows from Theorem 4.6 and Lemma 4.7 that G contains a subgraph T that is isomorphic to a subdivision of the infinite binary tree. A ray of T is *proper* if its vertices are linearly ordered by $<_T$. A *fork* in T is the graph consisting of two proper rays of T that meet in exactly one vertex, the *root* of the fork. For a binary sequence α, let t_α denote the vertex of T that corresponds to the vertex α of T^ω, and let ρ_α denote the ray of T containing all vertices t_γ for which γ consists of α followed by nothing but zeros.

For each binary sequence α of length exceeding one, let G_α denote the subgraph of $G'_{\mathbb{Z}\times\mathbb{Z}}$ induced by the vertices (γ,β) for which $\gamma \preccurlyeq \alpha$. We shall proceed by induction on the set $[2^{<\omega}]-\{\emptyset,0,1\}$ ordered by the relation $\preccurlyeq$ to define a sequence of isomorphisms f_{00}, f_{01}, $f_{10},\ldots$ such that, for each finite binary sequence α of length exceeding one, the following hold.

(1) f_α is an isomorphism from G_α to a minor of G.

(2) f_α a restriction of f_{α^+}.

(3) For every vertex of $G'_{\mathbb{Z}\times\mathbb{Z}}$ that has the form (α,β), there is a fork F in T such that the set $V(F)\cap V(f_\alpha(G_\alpha))$ consists of the root r of F, and $r\in V(f_\alpha((\alpha,\beta)))$.

(4) Let n denote the length of α. Then there is a set $\mathscr{R}_n$ of n pairwise-disjoint proper rays of T such that, for all $\beta\in[2^n]$, the image of the vertex (α,β) under f_α meets exactly one of the rays ρ of R_n, and the intersection $f_\alpha((\alpha,\beta))\cap\rho$ is a path.

Let $\mathscr{R}_2=\{\rho_{00},\rho_{01},\rho_{10},\rho_{11}\}$. Clearly, all rays in $\mathscr{R}_2$ are pairwise disjoint. Since G is round, it has a cycle C_{00} that collates $\mathscr{R}_2$. Let $\beta\in\{00,01,10,11\}$. As C_{00} is finite and G is locally finite, there is a fork F_β in T that is disjoint from C_{00}, and whose root r_β is such that $C_{00}<_{\rho_\beta} r_\beta$. Let $f_{00}(00,\beta)$ be the minimal path of ρ_β that contains $C_{00}\cap\rho_\beta$ and r_β. It is clear that (1)–(4) hold for $\alpha=00$.

Inductively, suppose that f_γ has been defined so that (1)–(4) hold for $\alpha=\gamma$. We shall consider two cases depending on the number of entries of γ^+.

Suppose first that the length of γ^+ equals the length of γ and denote this common length by n. Since G is round, there is a cycle C_{γ^+} in G that collates $\mathscr{R}_n$. Let $\beta\in[2^n]$ and let ρ_β be the ray from $\mathscr{R}_n$ that meets $f_\gamma((\gamma,\beta))$. Since C_{γ^+} is finite and G is locally finite, there is a fork F_β in T that is disjoint from C_{γ^+} and whose root r_β is such that $C_{\gamma^+}<_{\rho_\beta} r_\beta$. For all $x\in V(G_\gamma)\cup E(G_\gamma)$, let $f_{\gamma^+}(x)=f_\gamma(x)$, and, for all $\beta\in[2^n]$, let $f_{\gamma^+}((\gamma,\beta))$ to be the minimal path of ρ_β that contains $C_{\gamma^+}\cap\rho_\beta$ and r_β. Let B_γ be the subgraph of $G'_{\mathbb{Z}\times\mathbb{Z}}$ that is induced by the vertices of the form (γ,β) for all $\beta\in[2^n]$. Observe that, by the inductive application of (4), the image of B_γ under f_γ is a cycle collating $\mathscr{R}_n$. Now, C_{γ^+} collates R_n as well, and, by Lemma 2.4,

$\mathrm{tr}_{\mathscr{R}_n}(C_{\gamma^+}) = \mathrm{tr}_{\mathscr{R}_n}(f_\gamma(B_\gamma))$. Thus, it is easy to define f_{γ^+} on the edges of $G_{\gamma^+}\backslash G_\gamma$ so that (1)–(4) hold with $\alpha = \gamma^+$.

Now we may assume that the length of γ^+ exceeds the length of γ. For each vertex of $G'_{\mathbb{Z}\times\mathbb{Z}}$ that has the form (γ, β), let F_β be the fork as described in (4), and let r_β be the root of F_β. Observe that the graph $F_\beta - \{r_\beta\}$ consists of two disjoint rays. Denote these rays $\rho_{\beta+0}$ and $\rho_{\beta+1}$. Let $\mathscr{R}_{n+1} = \bigcup_{\beta\in[2^n]}\{\rho_{\beta+0}, \rho_{\beta+1}\}$. It is clear that $\mathscr{R}_{n+1}$ is a set of 2^{n+1} rays each two of which are disjoint. By (C3), there is a cycle C_{γ^+} in $G - f_\gamma(G_\gamma)$ that collates $\mathscr{R}_{n+1}$. It is now easy to use the ideas of the previous paragraph to extend f_γ to a function f_{γ^+} that is an isomorphism from G_{γ^+} to a minor of $G'_{\mathbb{Z}\times\mathbb{Z}}$ such that (1)–(4) hold with $\alpha = \gamma^+$. This completes the proof of Lemma 5.2. □

Proof of Lemma 5.3. This proof is similar to the proof of Lemma 4.3. Recall from that proof that, for a finite subgraph A of G, the union of A with all its finite bridges in G is denoted by $\overline{A}$. Let A_0 be an arbitrarily chosen finite subgraph of G on at least three vertices. Follow the inductive step of the definition of the A_i's from the proof of Lemma 4.3 by letting A_i be the subgraph of G induced by the vertices whose distance from $\overline{A_{i-1}}$ is less than two. Let $B_0 = A_0$, and, for a positive integer i, let $B_i = A_i\backslash\overline{A_{i-1}}$. Let $V_i = V(B_{i-1}) \cap V(B_i)$. Then all of (1)–(4) from the proof of Lemma 4.3 hold.

It follows from (3) that all vertices in V_1 lie in the boundary of the same face φ of B_0. Hence, by adding edges to B_0 if necessary, we form a plane graph B'_0 that has a cycle C_1 that bounds a face and contains all elements of V_1.

Now let i be a positive integer. From (4) we conclude that there is a face φ of B_i whose boundary contains all elements of V_i. Similarly, there is a face φ' of B_i whose boundary contains all elements of V_{i+1}. Again, by adding edges if necessary, we form a plane graph B'_i that has two disjoint cycles C_i and C_{i+1} such that each bounds a face of B' and $V(C_i) \supseteq V_i$ and $V(C_{i+1}) \supseteq V_{i+1}$.

Let $h_\emptyset$ denote the trivial isomorphism between null graphs. Then $(B'_0, (\emptyset, h_\emptyset), C_1)$ is a round segment. Hence, by Theorem 3.3, there is an integer n_0 and an isomorphism f_0 from B'_0 from B'_0 to a minor of Θ_{n_0} that agrees with $(B'_0, (\emptyset, h_\emptyset), C_1)$.

Inductively, suppose that i is a positive integer, $n_0, n_1, \ldots, n_{i-1}$ is an increasing sequence of integers, $(B'_{i-1}, (C_{i-1}, h_{i-1}), C_i)$ is a round segment, and f_{i-1} is an isomorphism from B'_{i-1} to a minor of $\Theta^0_{n_{i-1}}$ that agrees with $(B'_{i-1}, (C_{i-1}, h_{i-1}), C_i)$. Suppose v is a vertex of P_i. Then the set $V(f_{i-1}(v))$ contains a vertex $(\mathbf{1}, \beta_v)$ for some $\beta \in [2^{n_{i-1}}]$. Define the isomorphism h_i from C_i to a minor of $\Theta^0_{n_{i-1}}$ by letting $h_i(v) = (\mathbf{0}, \beta_v)$. Then $(B'_i, (C_i, h_i), P_{i+1})$ is a round segment, and by Theorem 3.3, there is an integer n_i exceeding n_{i-1} and an isomorphism from B'_i to a minor of Θ_{n_i} that agrees with $(B'_i, (C_i, h_i), P_{i+1})$.

For two binary sequences α and β such that α is an initial segment of β, let $P_{\alpha\beta}$ be the path of $G'_{\mathbb{Z}\times\mathbb{N}}$ defined as in the proof of Lemma 4.3. As $G'_{\mathbb{Z}\times\mathbb{N}}$ is a subgraph of $G'_{\mathbb{Z}\times\mathbb{Z}}$, each path $P_{\alpha\beta}$ is also a subgraph of $G'_{\mathbb{Z}\times\mathbb{Z}}$.

Upon following the last two paragraphs of the proof of Lemma 4.3 with $P_{\alpha\beta}$ replaced by $C_{\alpha\beta}$, we obtain an isomorphism f from G to a minor of $G'_{\mathbb{Z}\times\mathbb{Z}}$ that satisfies the conclusion of Lemma 5.3. □

6. Subgraphs of the full grid

In this section we shall investigate subgraphs of $G_{\mathbb{Z}\times\mathbb{Z}}$. Two edges e and f of $G_{\mathbb{Z}\times\mathbb{Z}}$ are *linked* if there is a face of $G_{\mathbb{Z}\times\mathbb{Z}}$ that has both e and f in its boundary. A subgraph H of $G_{\mathbb{Z}\times\mathbb{Z}}$ is *linked* if, for every two edges e and f of H, there is a sequence $e=e_0,e_1,\ldots,e_n=f$ of edges of $G_{\mathbb{Z}\times\mathbb{Z}}$ such that e_i is linked e_{i-1} for all $i\in\{1,2,\ldots,n\}$. Observe that there is a close relationship between linked subgraphs of $G_{\mathbb{Z}\times\mathbb{Z}}$ and connected duals of subgraphs of $G_{\mathbb{Z}\times\mathbb{Z}}$. We shall not pursue this relationship, however, as it will not be needed here. A *link-component* of a subgraph H of $G_{\mathbb{Z}\times\mathbb{Z}}$ is a maximal subgraph of H that is linked. The remainder of this section is devoted to proving the following two lemmas.

Lemma 6.1. *If H is an infinite linked subgraph of $G_{\mathbb{Z}\times\mathbb{Z}}$ such that $G_{\mathbb{Z}\times\mathbb{N}}\leqslant_{\mathrm{m}} G_{\mathbb{Z}\times\mathbb{N}}\backslash H$, then $G_{\mathbb{Z}\times\mathbb{Z}}\backslash H\leqslant_{\mathrm{m}} G_{\mathbb{Z}\times\mathbb{N}}$.*

Lemma 6.2. *If H is a subgraph of $G_{\mathbb{Z}\times\mathbb{Z}}$ such that all link-components of H are finite and $G_{\mathbb{Z}\times\mathbb{Z}}\backslash H$ contains infinitely many pairwise disjoint rays, then $G_{\mathbb{Z}\times\mathbb{Z}}\leqslant_{\mathrm{m}} G_{\mathbb{Z}\times\mathbb{Z}}\backslash H$.*

As preparation for proving Lemmas 6.1 and 6.2, we state and prove two easy lemmas about link graphs. For a cycle C of $G_{\mathbb{Z}\times\mathbb{Z}}$, the number of faces of $G_{\mathbb{Z}\times\mathbb{Z}}$ contained in the finite region of the plane cut off by C will be denoted by $a(C)$. Suppose H is a finite linked subgraph of $G_{\mathbb{Z}\times\mathbb{Z}}$. Let $\mathscr{C}(H)$ be the set of cycles C of $G_{\mathbb{Z}\times\mathbb{Z}}\backslash H$ such that all vertices of H lie on C or in the finite region of the plane cut off by C. A cycle C_0 in $\mathscr{C}(H)$ *surrounds* H if $a(C_0)\leqslant a(C)$ for every cycle C in $\mathscr{C}(H)$.

Lemma 6.3. *Let L be a subgraph of $G_{\mathbb{Z}\times\mathbb{Z}}$ that is a cycle or a two-way-infinite path. Then L induces a decomposition of $G_{\mathbb{Z}\times\mathbb{Z}}$ into pairwise-edge-disjoint union of three graphs: G',G'', and L such that G' and G'' are on the opposite sides of L. Suppose K is a subgraph of $G_{\mathbb{Z}\times\mathbb{Z}}\backslash L$ and K' in a link-component of K. Then K' meets at most one of $G'-L$ and $G''-L$.*

Proof. Suppose that K' meets both $G'-L$ and $G''-L$. Then there is a sequence $e_0,e_1,\ldots,e_n$ of edges of $G_{\mathbb{Z}\times\mathbb{Z}}\backslash L$ in which every two consecutive elements are linked, and such that e_0 is incident with a vertex of $G'-L$, and e_n is incident with a vertex of $G''-L$. Let k be the largest index such that e_k is incident with a vertex v' of $G'-L$. Clearly, $k<n$. Since e_{k+1} is not an edge of L, the choice of k implies that at least one of its vertices v'' lies in $G''-L$. Thus v' and v'' lie in the boundary of the same face of $G_{\mathbb{Z}\times\mathbb{Z}}$, and, at the same time, they lie on opposite sides of L; a contradiction. □

Lemma 6.4. *Let H be a finite linked subgraph of $G_{\mathbb{Z}\times\mathbb{Z}}$ and let C be a cycle of $G_{\mathbb{Z}\times\mathbb{Z}}\backslash H$ that surrounds H. Then, for every edge e of C, there is an edge f of H and a face φ of $G_{\mathbb{Z}\times\mathbb{Z}}$ such that e and f are in the boundary of φ.*

Proof. Suppose the lemma fails. Then there is an edge e of C that is not linked with any edges of H. Let φ denote the face of $G_{\mathbb{Z}\times\mathbb{Z}}$ that has e in its boundary and lies inside the finite region of the plane cut off by C. Let D denote the boundary of φ. Observe that, as H is linked and no edges of D are in H, it is impossible for $C\cap D$ to consist of two non-adjacent edges. Hence $C\cap D$ is a path. Construct a cycle C' from C by replacing $C\cap D$ by $D\backslash C$. Then C' is in $\mathscr{C}_H$ and $a(C')<a(c)$; a contradiction. □

Now we are ready to prove Lemmas 6.1 and 6.2.

Proof of Lemma 6.1. Observe that each edge of H is linked to finitely many and possibly no other edges of H. Hence, as H is linked and infinite, there is a sequence of edges $e_0, e_1, e_2, \dots$ of H such that

(1) *e_i and e_j are linked if and only if $|i-j|\leqslant 1$.*

Let K be the subgraph of $G_{\mathbb{Z}\times\mathbb{Z}}$ induced by the edges in $\{e_0, e_1, \dots\}$. Clearly, K is a subgraph of H, and hence, to prove Lemma 6.1, it suffices to show that

(2) $G_{\mathbb{Z}\times\mathbb{Z}}\backslash K \leqslant_{\mathrm{m}} G_{\mathbb{Z}\times\mathbb{N}}$.

It is clear that $G_{\mathbb{Z}\times\mathbb{Z}}\backslash K$ is planar and locally finite. We shall show that

(3) *$G_{\mathbb{Z}\times\mathbb{Z}}\backslash K$ has at most one end.*

Suppose ρ and σ are two rays in distinct ends of $G_{\mathbb{Z}\times\mathbb{Z}}\backslash K$. Let P be a path in $G_{\mathbb{Z}\times\mathbb{Z}}$ that joins ρ to σ. Then $\rho\cup\sigma\cup P$ has a subgraph L that is a two-way-infinite path. Now $G_{\mathbb{Z}\times\mathbb{Z}}$ can be represented as the union of three edge-disjoint graphs, G_1, G_2, and L, such that G_1 and G_2 are on opposite sides of L. Clearly, each of G_1 and G_2 contains infinitely many pairwise-disjoint paths each of which joins ρ to σ. As ρ and σ are in different rays of $G_{\mathbb{Z}\times\mathbb{Z}}\backslash K$, the graph K must meet all but finitely many of such $\rho\sigma$-paths. Hence, for infinitely many indices n, the edges e_n and e_{n+1} are on opposite sides of L. Since P is finite, there is an index n_0 such that none of the edges e_{n_0} and e_{n_0+1} is incident with vertices of P, yet e_{n_0} and e_{n_0+1} are on opposite sides of L. This is clearly impossible, and hence (3) follows.

Next we show that

(4) *$G_{\mathbb{Z}\times\mathbb{Z}}\backslash K$ is connected.*

Suppose not. Then, from (3), it follows that $G_{\mathbb{Z}\times\mathbb{Z}}\backslash K$ has a finite component G. Let E be the set of edges of K that are incident with vertices of G. The plane embedding of $G_{\mathbb{Z}\times\mathbb{Z}}$ induces a cyclic order on the elements of E with two successive elements in this order lying in the boundary of the same face. We obtain a contradiction to (1) as, clearly, E has more than two elements.

The next statement is the last we need in order to conclude that $G_{\mathbb{Z}\times\mathbb{Z}}\backslash K$ is flat.

(5) *One of the following holds*:
   (i) *$G_{\mathbb{Z}\times\mathbb{Z}}\backslash K$ fails to have three pairwise-disjoint rays*; *or*
   (ii) *there is a finite subgraph L of $G_{\mathbb{Z}\times\mathbb{Z}}\backslash K$ and a set $\mathscr{R}$ of three pairwise-disjoint rays of $G_{\mathbb{Z}\times\mathbb{Z}}\backslash K$ such that no cycle of $(G_{\mathbb{Z}\times\mathbb{Z}}\backslash K)-L$ collates $\mathscr{R}$.*

Suppose neither (i) nor (ii) holds. Then there is a set $\mathscr{R}$ of three pairwise-disjoint rays of $G_{\mathbb{Z}\times\mathbb{Z}}\backslash K$. Let L be a connected finite subgraph of $G_{\mathbb{Z}\times\mathbb{Z}}\backslash K$ that meets all rays of $\mathscr{R}$ and both endvertices of the edge e_0 of K. Then $(G_{\mathbb{Z}\times\mathbb{Z}}\backslash K) - L$ contains a cycle C that collates $\mathscr{R}$.

Observe now that, from the definition of K, it follows that there is a sequence $\varphi_1, \varphi_2, \varphi_3, \ldots$ of faces of $G_{\mathbb{Z}\times\mathbb{Z}}$ such that

(6) *for all positive integers i, the only edges of K in the boundary of φ_i are e_{i-1} and e_i; and*

(7) *if an edge e of $G_{\mathbb{Z}\times\mathbb{Z}}$ is in the boundaries of φ_i and φ_{i+1} for some integer i, then e is an edge of K.*

Construct a new graph $G'_{\mathbb{Z}\times\mathbb{Z}}$ from $G_{\mathbb{Z}\times\mathbb{Z}}$ by subdividing each edge e_i of K with a new vertex v_i, and then, for all positive integers i, joining the vertices v_{i-1} and v_i with a new edge f_i across the face φ_i. Let ρ' be the ray induced by the edges $f_1, f_2, f_3, \ldots$. Then $\mathscr{R} \cup \{\rho'\}$ is a set of four pairwise-disjoint rays, all of which are in the same end of $G'_{\mathbb{Z}\times\mathbb{Z}}$. Let L' be the subgraph of $G'_{\mathbb{Z}\times\mathbb{Z}}$ that is induced by L together with the two new edges obtained by subdividing e_0. Then L' is finite and connected. Note that $G'_{\mathbb{Z}\times\mathbb{Z}}$ is planar and locally finite, and C is a subgraph of $G'_{\mathbb{Z}\times\mathbb{Z}}$ that meets three elements of $\mathscr{R} \cup \{\rho'\}$. Thus, by Lemma 2.5, C also meets ρ'; a contradiction. Hence (5) holds.

In (3)–(5), we established that $G_{\mathbb{Z}\times\mathbb{Z}}\backslash K$ is flat. Hence, by Lemma 4.2, $G_{\mathbb{Z}\times\mathbb{Z}}\backslash K \leqslant_{\mathrm{m}} G_{\mathbb{Z}\times\mathbb{N}}$. As K is a subgraph of H, the theorem holds. □

Proof of Lemma 6.2. We shall show that

(1) *$G_{\mathbb{Z}\times\mathbb{Z}}\backslash H$ satisfies (C3).*

Let $\mathscr{R}$ be a set of pairwise-disjoint rays of $G_{\mathbb{Z}\times\mathbb{Z}}\backslash H$ such that $3 \leqslant |\mathscr{R}| < \infty$. Let K be a finite subgraph of $G_{\mathbb{Z}\times\mathbb{Z}}\backslash H$. Let K' be a connected subgraph of $G_{\mathbb{Z}\times\mathbb{Z}}$ that contains K and meets all rays in $\mathscr{R}$. Let $\mathscr{C}$ be the set of cycles of $G_{\mathbb{Z}\times\mathbb{Z}} - K'$ that collate $\mathscr{R}$. Since $G_{\mathbb{Z}\times\mathbb{Z}}$ is round, $\mathscr{C}$ is non-empty. For each cycle C is $\mathscr{C}$, let $r(C)$ denote the number of link-components of H whose edges are in C. Let C_0 be a cycle from $\mathscr{C}$ such that $r(C_0) \leqslant r(C)$ for all C in $\mathscr{C}$. We shall prove that

(2) *C_0 is contained in $(G_{\mathbb{Z}\times\mathbb{Z}}\backslash H) - K$.*

Suppose (2) fails. Then there is a link-component H_0 of H some of whose edges are in C_0. Recall that K' is connected and meets all of the rays in $\mathscr{R}$. Thus it follows from Lemma 6.3 that H_0 meets at most two rays ρ_1 and ρ_2 in $\mathscr{R}$, and that ρ_1 and ρ_2 are consecutive in the cyclic order induced on $\mathscr{R}$ by the plane embedding of $G_{\mathbb{Z}\times\mathbb{Z}}$. Consequently, H_0 does not contain C_0. Let $P = C_0\backslash H_0$. Then P is a disjoint union of paths each of which has both endvertices in the cycle D that surrounds H_0. Since C_0 avoids K' and collates $\mathscr{R}$, there is a connected component of P that collates $\mathscr{R} - \{\rho_1, \rho_2\}$. Thus the graph $D_0 \cup P \cup \rho_1 \cup \rho_2$ contains a cycle that collates $\mathscr{R}$. From among all such cycles, let C_1 be such that $a(C_1)$ is as large as possible. It is clear from the definition of K' and C_1 that C_1 avoids K. Suppose that an edge e of C_1 is in a link component H_1 of H. Then, clearly, e is an edge of D. Thus $H_1 \neq H_0$ and, by Lemma 6.4, there is an edge f of H_0 and a face φ of $G_{\mathbb{Z}\times\mathbb{Z}}$ such that e and f are in the boundary of φ; a contradiction. Hence C_1 avoids H and thus it is in $\mathscr{C}$. But then $r(C_1) < r(C_0)$, which contradicts the choice of C_0 and thus proves (2). The lemma follows. □

7. Proof of Theorem 1.4

In this section, we present the formal proof of Theorem 1.4. In the proof, we shall employ the concept of plane duals. For a plane graph G, its plane dual will be denoted by G^*. The following two lemmas are well-known results on graphs.

Lemma 7.1. *Let G be a plane graph and let E be a subset of the edge set of G such that $G \backslash E$ is isomorphic to a minor of a 3-connected graph K. Then G^*/E is isomorphic to a minor of K^*.*

Lemma 7.2. *Let G be a plane graph and let E be a subset of the edge set of G such that $G \backslash E$ has a minor isomorphic to a 3-connected graph K. Then G^*/E has a minor isomorphic to K^*.*

Proof of Theorem 1.4. Observe that the full grid is self-dual, and while the half-grid is not, it is minor-equivalent to its dual. It follows from Propositions 1.9, 1.10 and Theorems 4.1 and 5.1 that

(1) $G_{\mathbb{Z}\times\mathbb{N}} <_{\mathrm{m}} G_{\mathbb{Z}\times\mathbb{Z}}$.

Let K be a graph such that $G_{\mathbb{Z}\times\mathbb{N}} \leqslant_{\mathrm{m}} K \leqslant_{\mathrm{m}} G_{\mathbb{Z}\times\mathbb{Z}}$. To complete the proof of Theorem 1.4, we need to show that

(2) *either* $K \leqslant_{\mathrm{m}} G_{\mathbb{Z}\times\mathbb{N}}$ *or* $G_{\mathbb{Z}\times\mathbb{Z}} \leqslant_{\mathrm{m}} K$.

Let E and F denote sets of edges of $G_{\mathbb{Z}\times\mathbb{Z}}$ such that K is isomorphic to $G_{\mathbb{Z}\times\mathbb{Z}} \backslash E/F$. If some link-component of E is infinite, then, by Lemma 6.1, $G_{\mathbb{Z}\times\mathbb{Z}} \backslash E \leqslant_{\mathrm{m}} G_{\mathbb{Z}\times\mathbb{N}}$, and consequently $K \leqslant_{\mathrm{m}} G_{\mathbb{Z}\times\mathbb{N}}$. Suppose now that F has an infinite link-component. Then $(G_{\mathbb{Z}\times\mathbb{Z}} \backslash F) \leqslant_{\mathrm{m}} G_{\mathbb{Z}\times\mathbb{N}}$, and, as $G_{\mathbb{Z}\times\mathbb{N}}$ is 3-connected and minor-equivalent to its plane dual, we conclude from Lemma 7.1 that $G^*_{\mathbb{Z}\times\mathbb{Z}}/F \leqslant_{\mathrm{m}} G_{\mathbb{Z}\times\mathbb{N}}$. Since $G^*_{\mathbb{Z}\times\mathbb{N}}$ is isomorphic to $G_{\mathbb{Z}\times\mathbb{Z}}$, we have $K \leqslant_{\mathrm{m}} (G_{\mathbb{Z}\times\mathbb{Z}}/F) \leqslant_{\mathrm{m}} G_{\mathbb{Z}\times\mathbb{N}}$.

Now we may assume that all link-components of E and all link-components of F are finite. Let $L = G_{\mathbb{Z}\times\mathbb{Z}} \backslash E$. Then, by Lemma 6.2, $G_{\mathbb{Z}\times\mathbb{Z}} \leqslant_{\mathrm{m}} L$, and clearly

(3) $(G_{\mathbb{Z}\times\mathbb{Z}}/F) \leqslant_{\mathrm{m}} (L/F) = K$.

We apply Lemma 6.2 to conclude that $G_{\mathbb{Z}\times\mathbb{Z}} \leqslant_{\mathrm{m}} (G_{\mathbb{Z}\times\mathbb{Z}} \backslash F)$. Upon applying the fact that $G_{\mathbb{Z}\times\mathbb{Z}}$ is 3-connected and isomorphic to its plane dual, we conclude from Lemma 7.2 that $G_{\mathbb{Z}\times\mathbb{Z}} \leqslant_{\mathrm{m}} (G_{\mathbb{Z}\times\mathbb{Z}}/F)$. This together with (3) implies that $G_{\mathbb{Z}\times\mathbb{Z}} \leqslant_{\mathrm{m}} K$, as required. □

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